

# THE HOMOTOPY SEQUENCE OF THE ALGEBRAIC FUNDAMENTAL GROUP

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ABSTRACT. In this paper, we will prove that the homotopy sequence of the algebraic fundamental group is exact if the base field is of characteristic 0.

## 1. INTRODUCTION

If  $f : X \rightarrow S$  is a separable proper surjective morphism with geometrically connected fibres between locally noetherian connected schemes,  $x \rightarrow X$  is a geometric point with image  $s \rightarrow S$ , Grothendieck shows in [SGA1, Exposé X, Corollaire 1.4] that one has a homotopy exact sequence for the étale fundamental group:

$$\pi_1^{\text{ét}}(\bar{X}_s, x) \rightarrow \pi_1^{\text{ét}}(X, x) \rightarrow \pi_1^{\text{ét}}(S, s) \rightarrow 1.$$

A similar case is that one can take  $X, Y$  to be two locally noetherian connected  $k$ -schemes with  $k = \bar{k}$  and suppose  $Y$  is proper over  $k$ , so if  $K$  is an algebraically closed field containing  $k$  and if we take a  $K$ -point  $z = (x, y) : \text{Spec}(K) \rightarrow X \times_k Y$ , then we get a canonical morphism of groups

$$\pi_1^{\text{ét}}(X \times_k Y, z) \rightarrow \pi_1^{\text{ét}}(X, x) \times \pi_1^{\text{ét}}(Y, y).$$

Again Grothendieck shows in [SGA1, Exposé X, Corollaire 1.7] that the canonical homomorphism is an isomorphism. This is called the Künneth formula for the étale fundamental group. If  $X \times_k Y$  admits a  $k$ -rational point then the Künneth formula is a direct consequence of the homotopy exact sequence.

If  $X$  is a smooth geometrically connected scheme over a field  $k$  with a rational point  $x \in X(k)$ , we can consider the category of  $O_X$ -coherent  $D_{X/k}$ -modules which we will denote by  $\text{Mod}_c(D_{X/k})$ . Now let  $\omega_x$  be the functor  $\text{Mod}_c(D_{X/k}) \rightarrow \text{Vec}_k$  sending any  $O_X$ -coherent  $D_{X/k}$ -module  $M$  to  $M|_x$ . One can check that the category  $\text{Mod}_c(D_{X/k})$  together with  $\omega_x$  is a neutral Tannakian category, then we define its Tannakian group  $\pi^{\text{alg}}(X, x)$  to be the algebraic fundamental group of  $(X, x)$ . One can check easily that our construction is functorial in  $(X, x)$ . Thus if we have a proper smooth map  $f : X \rightarrow S$  between two smooth connected schemes over a field  $k$  with geometrically connected fibres, and assume  $x \in X(k)$  and  $s \in S(k)$  such that  $f(x) = s$ , then we will get a sequence of maps:

$$\pi^{\text{alg}}(X_s, x) \rightarrow \pi^{\text{alg}}(X, x) \rightarrow \pi^{\text{alg}}(S, s) \rightarrow 1.$$

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**2.2. The settings.** Let  $f : X \rightarrow S$  be a smooth proper morphism with geometrically connected fibres between two smooth connected schemes of finite type over a field  $k$ ,  $s \in S(k)$  be a rational point,  $X_s$  be the fibre,  $x \in X(k)$  be a rational point lying above  $s$ , then by the functoriality of the algebraic fundamental group we get a sequence of affine group schemes

$$\pi^{\text{alg}}(X_s, x) \rightarrow \pi^{\text{alg}}(X, x) \rightarrow \pi^{\text{alg}}(S, s) \rightarrow 1,$$

which is called the homotopy sequence. We will show that the sequence is exact if  $k$  has characteristic 0 by checking the conditions provided in the above theorem.

### 3. THE HOMOTOPY EXACT SEQUENCE IN CHARACTERISTIC 0

In this section  $k$  is always a field of characteristic 0. In this case the category  $\mathbf{Mod}_c(D_{X/k})$  is the same as the category of vector bundles with flat connections, so in the following we will work purely in the category of vector bundles with flat connections and still use  $\mathbf{Mod}_c(D_{X/k})$  to denote this category.

#### 3.1. The conditions (a), (b) and the surjectivity.

**Theorem 3.1.** *Notations and assumptions being as in §2.2, then the homotopy sequence*

$$\pi^{\text{alg}}(X_s, x) \rightarrow \pi^{\text{alg}}(X, x) \rightarrow \pi^{\text{alg}}(S, s) \rightarrow 1$$

*is a complex, and the arrow  $\pi^{\text{alg}}(X, x) \rightarrow \pi^{\text{alg}}(S, s)$  is surjective.*

*Proof.* Since  $s \in S$  is a rational point, we know that any object in  $\mathbf{Mod}_c(D_{S/k})$  is trivial after pulling back to  $\mathbf{Mod}_c(D_{X_s/k})$ , thus the sequence is a complex. To see the right arrow is surjective, one has to show that the functor  $f^* : \mathbf{Mod}_c(D_{S/k}) \rightarrow \mathbf{Mod}_c(D_{X/k})$  is fully faithful and stable under taking subquotient.

The fact that  $f^*$  is fully faithful follows readily from the projection formula, so we only have to show that it is stable under taking subquotient. Suppose we have an object  $(E, \nabla_E) \in \mathbf{Mod}_c(D_{S/k})$ , and a subobject  $(F, \nabla_F) \hookrightarrow f^*(E, \nabla_E)$ . Then  $f_*F$  is a locally free sheaf of rank equal to that of  $F$ ,  $f^*f_*F \rightarrow F$  is an isomorphism, and the natural map  $f_*F \rightarrow E$  imbeds  $f_*F$  as a subbundle of  $E$  (locally split). This can be seen in the following way.

First of all, for any  $t \in S$   $F|_{X_t}$  is a free  $O_{X_t}$ -module. This is because  $f^*(E, \nabla_E)|_{X_t/\kappa(t)}$  is a trivial object in  $\mathbf{Mod}_c(D_{X_t/\kappa(t)})$ , but  $(F, \nabla_F)|_{X_t/\kappa(t)} \subseteq f^*(E, \nabla_E)|_{X_t/\kappa(t)}$ , thus  $(F, \nabla_F)|_{X_t/\kappa(t)}$  is also a trivial object, so  $F|_{X_t}$  is a free  $O_{X_t}$ -module. This tells us  $f_*F$  satisfies base change for any  $t \in S$ , hence is a vector bundle. Then the canonical map  $f^*f_*F \rightarrow F$  is an isomorphism over all the fibres of  $t \in S$ , so itself is an isomorphism. This finishes the proof of the above claim.

Now from the connection  $\nabla_F$ , we get a map:

$$f_*F \rightarrow f_*(F \otimes_{O_X} \Omega_{X/k}^1) \cong f_*(f^*f_*F \otimes_{O_X} \Omega_{X/k}^1) \cong f_*F \otimes_{O_S} f_*\Omega_{X/k}^1.$$

Since  $f : X \rightarrow S$  is smooth, the exact sequence

$$0 \rightarrow f^*\Omega_{S/k}^1 \rightarrow \Omega_{X/k}^1 \rightarrow \Omega_{X/S}^1 \rightarrow 0$$

locally splits. Hence we have an induced injection

$$f^*(E/f_*F) \otimes_{O_X} f^*\Omega_{S/k}^1 \hookrightarrow f^*(E/f_*F) \otimes_{O_X} \Omega_{X/k}^1,$$

which is just

$$E/f_*F \otimes_{O_S} \Omega_{S/k}^1 \hookrightarrow E/f_*F \otimes_{O_S} f_*\Omega_{X/k}^1$$

by applying  $f_*$  and the projection formula. Now look at the following commutative diagramme with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & f_*F \otimes_{O_S} \Omega_{S/k}^1 & \longrightarrow & E \otimes_{O_S} \Omega_{S/k}^1 & \longrightarrow & E/f_*F \otimes_{O_S} \Omega_{S/k}^1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & f_*F \otimes_{O_S} f_*\Omega_{X/k}^1 & \longrightarrow & E \otimes_{O_S} f_*\Omega_{X/k}^1 & \longrightarrow & E/f_*F \otimes_{O_S} f_*\Omega_{X/k}^1 & \longrightarrow & 0 \end{array}$$

Since  $f_*F$  maps to  $f_*F \otimes_{O_S} f_*\Omega_{X/k}^1$ , its image in  $E/f_*F \otimes_{O_S} f_*\Omega_{X/k}^1$  is trivial. Because  $E/f_*F \otimes_{O_S} \Omega_{S/k}^1 \hookrightarrow E/f_*F \otimes_{O_S} f_*\Omega_{X/k}^1$  is injective,  $f_*F \rightarrow E \otimes_{O_S} \Omega_{S/k}^1$  factors through  $f_*F \otimes_{O_S} \Omega_{S/k}^1$ . This proves that  $f_*F \subseteq E$  is equipped with a flat connection  $f_*\nabla_F$  which makes  $(f_*F, f_*\nabla_F)$  a subobject of  $(E, \nabla_E)$ . Clearly  $f^*(f_*F, f_*\nabla_F) \cong (F, \nabla_F)$  as subobjects of  $f^*(E, \nabla_E)$ . This finishes the proof.  $\square$

**Corollary 3.2.** *Notations and assumptions being as in §2.2, then for any object  $(E, \nabla_E) \in \text{Mod}_c(D_{X/k})$ , the natural map*

$$\phi : f^*H_{DM}^0(X/S, (E, \nabla_E)) = f^*f_*E^{\nabla_{X/S}} \rightarrow E$$

*is a horizontal with respect to the Gauss-Manin connection on the left (i.e. a morphism in  $\text{Mod}_c(D_{S/k})$ ). Furthermore this map is injective and imbeds  $f^*H_{DM}^0(X/S, (E, \nabla_E))$  as the maximal subobject of  $(E, \nabla_E)$  coming from  $S/k$  in the following sense:*

*If  $(M, \nabla_M) \subseteq (E, \nabla_E) \in \text{Mod}_c(D_{X/k})$  such that  $(M, \nabla_M) = f^*(N, \nabla_N)$  for some  $(N, \nabla_N) \in \text{Mod}_c(D_{S/k})$ , then the imbedding  $(M, \nabla_M) \subseteq (E, \nabla_E)$  factors through  $\phi$ .*

*Proof.* The fact that  $\phi$  is horizontal is from the definition of the Gauss-Manin connection. To show that it is injective one considers the kernel  $(K, \nabla_K)$  of the map. One has:

$$0 \rightarrow (K, \nabla_K) \rightarrow f^*H_{DM}^0(X/S, (E, \nabla_E)) \xrightarrow{\phi} (E, \nabla_E)$$

is exact. Since the functor  $H_{DM}^0(X/S, -)$  is left exact and  $H_{DM}^0(X/S, \phi)$  is an isomorphism, we have  $H_{DM}^0(X/S, (K, \nabla_K)) = 0$ . But by the theorem above, one has  $(K', \nabla_{K'}) \in \text{Mod}_c(D_{S/k})$  such that  $f^*(K', \nabla_{K'}) = (K, \nabla_K)$ . Thus as sheaves on  $S$ , one has  $0 = H_{DM}^0(X/S, (K, \nabla_K)) = H_{DM}^0(X/S, f^*(K', \nabla_{K'})) \cong K'$ . This shows that  $K = 0$ . Thus  $\phi$  is injective.

Suppose  $(M, \nabla_M) \subseteq (E, \nabla_E) \in \text{Mod}_c(D_{X/k})$  such that  $(M, \nabla_M) = f^*(N, \nabla_N)$  for some  $(N, \nabla_N) \in \text{Mod}_c(D_{S/k})$ , then  $N \cong f_*M^{\nabla_{X/S}} \hookrightarrow f_*E^{\nabla_{X/S}} = H_{DM}^0(X/S, (E, \nabla_E))$ , this shows  $M \hookrightarrow E$  factors through  $\phi$ .  $\square$

**Theorem 3.3.** *Notations and assumptions being as in §2.2, then for any  $(E, \nabla_E) \in \mathbf{Mod}_c(D_{X/k})$  the subobject*

$$(F, \nabla_F) := f^* H_{DM}^0(X/S, (E, \nabla_E)) \hookrightarrow (E, \nabla_E)$$

*has the restriction  $(F, \nabla_F)|_{X_s/k} \hookrightarrow (E, \nabla_E)|_{X_s/k}$  which gives the maximal trivial subobject of  $(E, \nabla_E)|_{X_s/k}$ . So in particular, our condition (a) and (b) are satisfied.*

*Proof.* Since the maximal trivial subobject of  $(E, \nabla_E)|_{X_s/k}$  is precisely

$$f^* H_{DM}^0(X_s/k, (E, \nabla_E)|_{X_s/k}) \hookrightarrow (E, \nabla_E)|_{X_s/k},$$

so the above theorem is just the base theorem for the Gauss-Manin Connection which was proved in [Katz][Section 8].  $\square$

**3.2. The condition (c) for a generic geometric point.** Now we come to check the condition (c) in our general criterion. Since we are not going to show the injectivity of the very left arrow, the condition (c) in our situation reads:

For  $\forall (E, \nabla_E) \in \mathbf{Mod}_c(D_{X/k})$  and any quotient  $(E, \nabla_E)|_{X_s/k} \twoheadrightarrow (F', \nabla_{F'}) \in \mathbf{Mod}_c(D_{X_s/k})$ ,  $\exists (F, \nabla_F) \in \mathbf{Mod}_c(D_{X/k})$  and an imbedding  $(F', \nabla_{F'}) \hookrightarrow (F, \nabla_F)|_{X_s/k} \in \mathbf{Mod}_c(D_{X_s/k})$ . Or equivalently, one can say (by taking dual) for  $\forall (E, \nabla_E) \in \mathbf{Mod}_c(D_{X/k})$  and any subobject  $(E, \nabla_E)|_{X_s/k} \hookrightarrow (F', \nabla_{F'}) \in \mathbf{Mod}_c(D_{X_s/k})$ ,  $\exists (F, \nabla_F) \in \mathbf{Mod}_c(D_{X/k})$  and a surjection  $(F', \nabla_{F'}) \twoheadrightarrow (F, \nabla_F)|_{X_s/k} \in \mathbf{Mod}_c(D_{X_s/k})$ .

This condition here is quite difficult to check, but since (a) and (b) are satisfied now it is equivalent to the exactness of the homotopy sequence. We will first prove this condition in a special case (for a generic geometric point) then we will show that if in this special case our condition is OK then the homotopy sequence is exact in general. Next we will place the settings for the generic geometric point.

**The Setup of §3.2:** Let  $f_0 : X_0 \rightarrow S_0$  be a smooth morphism between smooth geometrically connected schemes of finite type over a field  $k_0$  of characteristic 0. Let  $k$  be an algebraic extension of  $\kappa(S_0)$ ,  $S := S_0 \times_{k_0} k$ ,  $X := X_0 \times_{k_0} k$ ,  $f := f_0 \times_{k_0} k$ . Now we get a  $k$ -rational point  $s \in S$  which corresponds to the generic point of  $S_0$ . Let  $X_s$  be the fibre of  $s \in S(k)$ ,  $x \in X(k)$  such that  $f(x) = s$ .

**Proposition 3.4.** *If  $(E, \nabla_E) \in \mathbf{Mod}_c(D_{X/k})$ ,  $(F', \nabla_{F'}) \subseteq (E, \nabla_E)|_{X_s/k} \in \mathbf{Mod}_c(D_{X_s/k})$ , then  $\exists$  a non-trivial Zariski open  $U_0 \subseteq S_0$  and an object  $(F_0, \nabla_{F_0}) \in \mathbf{Mod}_c(D_{f_0^{-1}(U_0)/k_0})$  with a surjection  $(F_0, \nabla_{F_0})|_{X_s/k} \twoheadrightarrow (F', \nabla_{F'})$ .*

*Proof.* According to Lemma 3.5 below, we have a non-trivial Zariski open  $U_0 \subseteq S_0$  and a finite étale covering  $T_0 \rightarrow U_0$  with  $\kappa(T_0) \subseteq k$  such that  $(E, \nabla_E) \in \mathbf{Mod}_c(D_{X/k})$  is defined over  $\mathbf{Mod}_c(D_{X_0 \times_{k_0} T_0/T_0})$ . We may assume  $U_0 = S_0$  and let  $(E_0, \nabla_{E_0}) \in \mathbf{Mod}_c(D_{X_0 \times_{k_0} T_0/T_0})$  be the object such that  $\rho^*(E_0, \nabla_{E_0}) \cong (E, \nabla_E)$  where  $\rho : X = X_0 \times_{k_0} k \rightarrow X_0 \times_{k_0} T_0$ . Let  $\alpha : T_0 \hookrightarrow S_0 \times_{k_0} T_0$  be the graph of  $T_0 \rightarrow S_0$  and  $\beta : X_{0T_0} \hookrightarrow X_0 \times_{k_0} T_0$  be the pull back of

the graph:

$$\begin{array}{ccc} X_{0T_0} & \xrightarrow{\beta} & X_0 \times_{k_0} T_0 \\ \downarrow & & \downarrow \\ T_0 & \xrightarrow{\alpha} & S_0 \times_{k_0} T_0 \end{array}$$

Then the pull back  $\beta^*(E_0, \nabla_{E_0}) \in \mathbf{Mod}_c(D_{X_{0T_0}/T_0})$  is actually defined over  $\mathbf{Mod}_c(D_{X_0T_0/k_0})$ . In fact, we have the following commutative diagram:

$$\begin{array}{ccccc} X_{0T_0} & \xrightarrow{\beta} & X_0 \times_{k_0} T_0 & \xrightarrow{p} & X_0 \\ \downarrow & & \downarrow & & \downarrow \\ T_0 & \xlongequal{\quad} & T_0 & \longrightarrow & k_0 \end{array}$$

Thus we have maps

$$\beta^* \Omega_{X_0 \times_{k_0} T_0/T_0}^1 \cong \beta^* p^* \Omega_{X_0/k_0}^1 \rightarrow \Omega_{X_{0T_0}/k_0}^1.$$

Note that the last arrow in the above sequence is actually coming from the following commutative diagramme:

$$\begin{array}{ccc} X_{0,T_0} & \xrightarrow{p \circ \beta} & X_0 \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k_0) & \xlongequal{\quad} & \mathrm{Spec}(k_0) \end{array}$$

This indeed extends our connection

$$\nabla_{E_0} : E_0 \rightarrow E_0 \otimes_{\mathcal{O}_{X_0 \times_{S_0} T_0}} \Omega_{X_0 \times_{k_0} T_0/T_0}^1 \cong E_0 \otimes_{\mathcal{O}_{X_0 \times_{S_0} T_0}} p^* \Omega_{X_0/k_0}^1$$

to the connection

$$\beta^* \nabla_{E_0} : \beta^* E_0 \rightarrow \beta^* E_0 \otimes_{\mathcal{O}_{X_{0T_0}}} \Omega_{X_{0T_0}/k_0}^1.$$

Let  $\lambda : X_{0T_0} \rightarrow X_{0S_0} \cong X_0$ . Since  $T_0 \rightarrow S_0$  is finite étale, we have  $\lambda_* \beta^*(E_0, \nabla_{E_0}) \in \mathbf{Mod}_c(D_{X_0/k_0})$  and a surjection

$$\lambda^* \lambda_* \beta^*(E_0, \nabla_{E_0}) \twoheadrightarrow \beta^*(E_0, \nabla_{E_0}).$$

From the Cartesian diagrams

$$\begin{array}{ccccccc} X_s & \xrightarrow{\iota} & X_{0T_0} & \xrightarrow{\beta} & X_0 \times_{k_0} T_0 & \longrightarrow & X_0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ k & \longrightarrow & T_0 & \longrightarrow & S_0 \times_{k_0} T_0 & \longrightarrow & S_0 \end{array}$$

we know that if we pull back  $\beta^*(E_0, \nabla_{E_0})$  along  $\iota$  then we get  $(E, \nabla_E)|_{X_s/k}$ . Now let  $(F'', \nabla_{F''})$  be the inverse image of  $(F', \nabla_{F'})$  under the map

$$\lambda^* \lambda_* \beta^*(E_0, \nabla_{E_0})|_{X_s/k} \twoheadrightarrow \beta^*(E_0, \nabla_{E_0})|_{X_s/k} = (E, \nabla_E)|_{X_s/k}.$$

According to our lemma 3.6 below, there exists non-trivial Zariski open  $U_0 \subseteq S_0$  and  $(F_0, \nabla_{F_0}) \in \mathbf{Mod}_c(D_{f_0^{-1}(U_0)/k_0})$  with a surjection  $(F_0, \nabla_{F_0})|_{X_s/k} \twoheadrightarrow (F'', \nabla_{F''}) \twoheadrightarrow (F', \nabla_{F'}) \in \mathbf{Mod}_c(D_{X_s/k})$ . This completes the proof.  $\square$

**Lemma 3.5.** (The notations and conventions in this lemma are independent) *Let  $f : X \rightarrow S$  be a smooth morphism between two integral noetherian schemes. Let  $s \in S$  be the generic point,  $\kappa(s) \subseteq k$  be a separable algebraic extension of fields,  $X_k$  be the generic fibre (corresponding to  $\mathrm{Spec}(k) \hookrightarrow S$ ). Then for any object  $(F, \nabla_F) \in \mathbf{Mod}_c(D_{X_k/k})$  with  $F$  a vector bundle, there exists a non-empty open subset  $U \subseteq S$ , an integral finite étale covering  $T \rightarrow U$  and an object  $(E, \nabla_E) \in \mathbf{Mod}_c(D_{X \times_S T/T})$  which satisfy (1) the function field of  $T$  is contained in  $k$ ; (2)  $(F, \nabla_F) \cong (E, \nabla_E)_{X_k/k}$ .*

*Proof.* Let  $\phi : X_k \rightarrow X$  be the canonical imbedding of the generic fibre and assume  $S = \mathrm{Spec}(R)$ . Then we get a surjection  $\phi^* \phi_* F \twoheadrightarrow F$ . Since  $\phi_* F$  is the union of its coherent subsheaves, we find a coherent subsheaf  $M$  of  $\phi_* F$  with a surjection  $\phi^* M \twoheadrightarrow F$ . Suppose  $N \subseteq \phi^* M$  is the kernel of  $\phi^* M \rightarrow F$ . It is coherent since  $X$  is noetherian. Then we can collect finitely many elements  $\{x_0, \dots, x_n\}$  in  $k$  which are integral over  $R$  and a non-zero element  $f \in R$  such that  $N$  is defined over  $R_1 := R_f[x_0, \dots, x_n]$ . Thus  $F$  is defined over  $R_1$ . Let's say  $E_1$  is a coherent sheaf on  $X \times_R R_1$  such that  $\rho_1^* E_1 \cong F$ , where  $\rho_1 : X_k = X \times_R k \rightarrow X \times_R R_1$ . Since the problem is local for  $S$ , and  $F$  is locally free, we may assume  $E_1$  is locally free. Then the map

$$E_1 \otimes_{O_{X \times_R R_1}} \Omega_{X \times_R R_1/R_1}^1 \rightarrow \rho_{1*} \rho_1^* (E_1 \otimes_{O_{X \times_R R_1}} \Omega_{X \times_R R_1/R_1}^1)$$

is injective. Since the  $k$ -linear map

$$\nabla_F : F \rightarrow F \otimes_{O_{X_k}} \Omega_{X_k/k}^1$$

can be seen as a map

$$\rho_1^* E_1 \rightarrow \rho_1^* (E_1 \otimes_{O_{X \times_R R_1}} \Omega_{X \times_R R_1/R_1}^1),$$

we can collect finite many elements  $\{y_0, \dots, y_n\}$  in  $k$  which are integral over  $R$  and a non-zero element  $g \in R$  such that  $\nabla_F$  is defined over  $R_2 = (R_1)_g[y_0, \dots, y_n]$  and is still a flat connection. Thus we have found  $T_2 := \mathrm{Spec} R_2$  and  $(E_2, \nabla_{E_2}) \in FConn(X \times_S T_2/T_2)$  such that  $\rho_2^* (E_2, \nabla_{E_2}) \cong (F, \nabla_F)$  (where  $\rho_2 : X \times_S k \rightarrow X \times_S T_2$ ) and the generic point of  $T_2$  is a finite field extension of  $\kappa(s)$ . Now the map  $T_2 \rightarrow S$  which is finite onto its image is étale at the generic point of  $T_2$ , thus we get a non-empty open sub  $T$  of  $T_2$  such that  $T$  is finite étale over some non-empty open  $U$  of  $S$ . This is precisely what we want.  $\square$

**Lemma 3.6.** *For any object  $(E, \nabla_E) \in \mathbf{Mod}_c(D_{X_0/k_0})$  and any imbedding  $(F', \nabla_{F'}) \hookrightarrow (E, \nabla_E)|_{X_s/k} \in \mathbf{Mod}_c(D_{X_s/k})$  there is a non-empty open  $U_0 \subseteq S_0$  and an object  $(F, \nabla_F) \in \mathbf{Mod}_c(D_{f^{-1}(U_0)/k_0})$  which admits a surjection*

$$(F, \nabla_F)|_{X_s/k} \twoheadrightarrow (F', \nabla_{F'}) \hookrightarrow (E, \nabla_E)|_{X_s/k} \in \mathbf{Mod}_c(D_{X_s/k}).$$

*Proof.* First suppose  $\kappa(S_0) \subseteq k$  is a trivial extension. Let  $r := \dim_{O_{X_s}}(F')$ . According to [EP][Theorem 5.10] we have a subobject  $(M, \nabla_M) \subseteq (E, \nabla_E)|_{X_s/k_0} \in \mathbf{Mod}_c(D_{X_s/k_0})$

with a surjection  $(M, \nabla_M)|_{X_s/k} \twoheadrightarrow \det(F', \nabla_{F'})$ . If we set  $(F_1, \nabla_{F_1}) := (M, \nabla_M) \otimes_{O_{X_s}} (\wedge^{r-1}(E, \nabla_E)|_{X_s/k_0})^\vee$ , then it is a subobject

$$(F_1, \nabla_{F_1}) \subseteq (E, \nabla_E)|_{X_s/k_0} \otimes_{O_{X_s}} (\wedge^{r-1}(E, \nabla_E)|_{X_s/k_0})^\vee \in \mathbf{Mod}_c(D_{X_s/k_0})$$

with a surjection

$$(F_1, \nabla_{F_1})|_{X_s/k} \twoheadrightarrow (F', \nabla_{F'}) \cong \det(F', \nabla_{F'}) \otimes_{O_{X_s}} (\wedge^{r-1}(F', \nabla_{F'}))^\vee.$$

Let  $u : X_s \rightarrow X_0$  be the canonical imbedding, then we take the inverse image of  $u_* F_1$  under the canonical map

$$E \otimes_{O_X} (\wedge^{r-1} E)^\vee \rightarrow u_* u^*(E \otimes_{O_X} (\wedge^{r-1} E)^\vee)$$

and denote it by  $F_2$ . One can check there is a non-empty open subscheme  $U_0 \subseteq S_0$  so that  $F_2$  is equipped with a flat connection on  $f^{-1}(U_0)/k_0$  and becomes a subobject

$$(F_2, \nabla_{F_2}) \subseteq ((E, \nabla_E) \otimes_{O_{X_0}} (\wedge^{r-1}(E, \nabla_E))^\vee)|_{f^{-1}(U_0)/k_0} \in \mathbf{Mod}_c(D_{f^{-1}(U_0)/k_0})$$

which satisfies  $(F_2, \nabla_{F_2})|_{X_s/k_0} \cong (F_1, \nabla_{F_1})$ . This finishes the special case.

Now suppose  $\kappa(S_0) \subseteq k$  is a non-trivial extension. It is clear that the map  $(F', \nabla_{F'}) \hookrightarrow (E, \nabla_E)|_{X_s/k}$  is defined over  $\mathbf{Mod}_c(D_{X_{k'}/k'})$  where  $k'$  is a finite extension of  $\kappa(S_0)$  and  $X_{k'} := X_0 \times_{S_0} k'$ . Thus we may assume  $k/\kappa(S_0)$  is finite. Then the map  $\alpha : X_s \rightarrow X_0 \times_{S_0} \kappa(S_0)$  is finite étale. So we get a surjection

$$\alpha^* \alpha_*(F', \nabla_{F'}) \twoheadrightarrow (F', \nabla_{F'}) \in \mathbf{Mod}_c(D_{X_s/k})$$

and an imbedding

$$\alpha_*(F', \nabla_{F'}) \hookrightarrow \alpha_*((E, \nabla_E)|_{X_s/k}) \in \mathbf{Mod}_c(D_{X_0 \times_{S_0} \kappa(S_0)/\kappa(S_0)}).$$

Thus it is enough to show that  $\alpha_*((E, \nabla_E)|_{X_s/k})$  is defined in  $\mathbf{Mod}_c(D_{f^{-1}(U_0)/k_0})$  with  $U_0 \subseteq S_0$  non-trivial Zariski open, since then we can apply the special case we discussed above to get a surjection on  $\alpha_*(F', \nabla_{F'})$  from some object in  $\mathbf{Mod}_c(D_{f^{-1}(U_0)/k_0})$ . Since the problem is local on  $S_0$  we may assume  $S_0 = \text{Spec}(R)$ . Then one can find a finite ring extension  $R \subseteq R' \subseteq k$  such that  $R'$  has quotient field  $k$  (ex. the integral closure of  $R$  in  $k$ ). Again because our problem is local on  $S_0$ , one may assume  $R'/R$  is finite étale. Let  $\beta : X'_0 := X_0 \times_{\text{Spec}(R)} \text{Spec}(R') \rightarrow X_0$ ,  $u : X_0 \times_{S_0} \kappa(S_0) \rightarrow X_0$ . Then  $u^* \beta_* \beta^*(E, \nabla_E) \cong \alpha_*((E, \nabla_E)|_{X_s/k})$ , but  $\beta_* \beta^*(E, \nabla_E) \in \mathbf{Mod}_c(D_{X_0/k_0})$ . This completes the proof.  $\square$

**Definition 3.7.** Let  $\mathbf{Mod}_c(D_{S/k}, s)$  be the category whose objects are of the form  $(U, M)$ , where  $U$  is an open subset of  $S$  containing  $s$  and  $M$  is a coherent sheaf on  $U$  with a flat connection  $\nabla_M$  on  $U/k$ , whose morphisms between two objects  $(U, M)$  and  $(U', M')$  are defined by

$$\text{Mor}((U, M), (U', M')) := \text{Hom}_{U \cap U'}((U, M)|_{U \cap U'}, (U', M')|_{U \cap U'}).$$

Let  $\mathbf{Mod}_c(D_{X/S/k}, f, s)$  be the category whose objects are of the form  $(U, M)$ , where  $U$  is an open subset of  $S$  containing  $s$  and  $M$  is a coherent sheaf on  $f^{-1}(U)$  with a flat connection  $\nabla_M$  on  $f^{-1}(U)/k$ , whose morphisms between two objects  $(U, M)$  and  $(U', M')$  are defined by

$$\text{Mor}((U, M), (U', M')) := \text{Hom}_{f^{-1}(U \cap U')}((U, M)|_{f^{-1}(U \cap U')}, (U', M')|_{f^{-1}(U \cap U')}).$$



**Proposition 3.8.** *Let  $X$  be a smooth geometrically connected scheme of finite type over a field  $k$  of characteristic 0,  $U \subseteq X$  be a dense open subscheme, then for any two objects  $(E, \nabla_E), (F, \nabla_F) \in \mathbf{Mod}_c(D_{X/k})$  and any morphism  $f_U : (E, \nabla_E)|_{U/k} \rightarrow (F, \nabla_F)|_{U/k} \in \mathbf{Mod}_c(D_{U/k})$ , we can uniquely extend  $f_U$  to a morphism*

$$f : (E, \nabla_E) \rightarrow (F, \nabla_F) \in \mathbf{Mod}_c(D_{X/k}).$$

*Proof.* The uniqueness is clear since if we have two extensions  $f$  and  $f'$  then the set of points of  $X$  on which  $f = f'$  is closed. By the lemma below we may assume  $f_U$  is an isomorphism.

Now suppose for any point  $x \in X \setminus U$  we can extend  $f_U$  to a neighborhood of  $x$ , then using Zorn's lemma we can extend  $f_U$  to a map on  $X$ . Hence the problem is local. We may assume  $X = \mathrm{Spec}(A)$  is a smooth integral  $k$ -algebra with an étale coordinate  $X \rightarrow \mathbb{A}_k^r$  ( $r = \dim X$ ), and

$$E = F = A^n := \underbrace{A \oplus \cdots \oplus A}_n,$$

and  $U = \mathrm{Spec}(A_f)$  with  $f$  non-zero in  $A$ . Let  $d : A^n \rightarrow A^n \otimes_A \Omega_{A/k}^1$  be the canonical connection (the  $n$ -th product of the trivial connections). Adding the  $A$ -linear map  $d - \nabla_E$  on both of the left and the right sides of the following commutative diagram :

$$\begin{array}{ccc} A_f^n & \xrightarrow{f_U} & A_f^n \\ \nabla_E \downarrow & & \downarrow \nabla_F \\ A_f^n \otimes_{A_f} \Omega_{A_f/k}^1 & \xrightarrow{f_U \otimes id} & A_f^n \otimes_{A_f} \Omega_{A_f/k}^1 \end{array}$$

we get a commutative diagram:

$$\begin{array}{ccc} A_f^n & \xrightarrow{f_U} & A_f^n \\ d \downarrow & & \downarrow d + (\nabla_F - \nabla_E) \\ A_f^n \otimes_{A_f} \Omega_{A_f/k}^1 & \xrightarrow{f_U \otimes id} & A_f^n \otimes_{A_f} \Omega_{A_f/k}^1 \end{array}.$$

We note that  $d + (\nabla_F - \nabla_E)$  is still a flat connection on  $A^n$ . Let  $\{e_i\}_{1 \leq i \leq n}$  be the canonical basis of  $A^n$  as a free  $A$ -module. From the commutative diagram one sees that the image of  $e_i$  under  $f_U : A_f^n \rightarrow A_f^n$  is a horization section of  $d + (\nabla_F - \nabla_E)$  on  $U$  for each  $i$ . It suffices to prove the fact that the restriction

$$H_{DM}^0(X, (E, d + (\nabla_F - \nabla_E))) \subseteq H_{DM}^0(U, (E_U, d + (\nabla_F - \nabla_E)))$$

is an isomorphism, since then the image  $f_U(e_i)$  is in  $A^n$  for each  $i$ .

Suppose  $I = (i_1, i_2, \dots, i_n)$  be a vector with entries in  $\Omega_{A/k}^1$  such that for any vector  $v = (v_1, v_2, \dots, v_n)$  in  $A^n$  we have

$$(\nabla_F - \nabla_E)(v) = (v_1 i_1, v_2 i_2, \dots, v_n i_n).$$

Let  $w = (w_1, w_2, \dots, w_n)$  be a vector in  $H_{DM}^0(U, (E_U, d + (\nabla_F - \nabla_E)))$ . Then we have

$$(d(w_1), d(w_2), \dots, d(w_n)) = -(w_1 i_1, w_2 i_2, \dots, w_n i_n).$$

Now if  $w_1 \notin A$ , then there is a codimension 1 prime ideal  $p \in X$  such that  $w_1 = a\pi^{-k}$  with  $\pi$  the uniformizer,  $a \in A_p$  invertible, and  $k > 0$ . Then we have

$$d(w_1) = \pi^{-k}d(a) + ka\pi^{-k-1}d(\pi).$$

But we from the second fundamental exact sequence

$$0 \rightarrow p/p^2 \rightarrow \Omega_{A_p/k}^1 \otimes_k A/p \rightarrow \Omega_{(A/p)/k}^1 \rightarrow 0$$

(this sequence is split exact because there is neighborhood of  $p$  in which  $A/p$  is smooth over  $k$ ) we know that  $d(\pi)$  could be extended to a basis of the free  $A_p$ -module  $\Omega_{A_p/k}^1$ . This tells us that  $\pi^k d(w_1)$  is not a regular differential 1-form in  $\Omega_{A_p/k}^1$ . This contradicts to the formulae  $d(w_1) = -w_1 i_1$  because  $i_1$  is a regular 1-form. Thus  $w \in A^n$  this is just what we want to show.  $\square$

**Lemma 3.9.** *Let  $X$  be a smooth  $k$ -scheme,  $U \subseteq X$  be an open subset,  $(E, \nabla_E) \in \mathbf{Mod}_c(D_{X/k})$  and suppose there is an injection*

$$(F', \nabla_{F'}) \hookrightarrow (E, \nabla_E)|_{U/k} \in \mathbf{Mod}_c(D_{U/k}).$$

*Then there exists a subobject  $(F, \nabla_F) \subseteq (E, \nabla_E) \in \mathbf{Mod}_c(D_{X/k})$  such that  $(F, \nabla_F)|_{U/k} = (F', \nabla_{F'})$  as subobjects of  $(E, \nabla_E)|_{U/k}$*

*Proof.* Let  $j : U \subseteq X$  be the inclusion. We take  $F$  to be the inverse image of  $j_* F'$  under the adjunction map  $E \rightarrow j_* j^* E$ . Then  $F$  is a coherent sheaf, and one checks easily that  $F \rightarrow E \xrightarrow{\nabla_E} E \times_{O_X} \Omega_{X/k}^1$  factors through  $F \times_{O_X} \Omega_{X/k}^1 \rightarrow E \times_{O_X} \Omega_{X/k}^1$ . Hence  $F$  is equipped with a connection  $\nabla_F$  and becomes a subobject of  $(E, \nabla_E)$ . Clearly  $(F, \nabla_F)|_{U/k}$  is equal to  $(F', \nabla_{F'})$  as subobjects (since  $F|_U$  is equal to  $F'$ ).  $\square$

The above Proposition and the above Lemma implies immediately the following:

**Lemma 3.10.** *The category  $\mathbf{Mod}_c(D_{S/k}, S)$  (resp.  $\mathbf{Mod}_c(D_{X/S/k}, f, s)$ ) is an abelian  $k$ -linear rigid tensor category equipped with an exact faithful  $k$ -linear tensor functor  $(M, U) \mapsto M|_s$  (resp.  $(M, U) \mapsto M|_x$ ). Thus it is a neutral Tannakian category, so we have a Tannakian group  $\hat{\pi}^{alg}(S, s)$  (resp.  $\hat{\pi}^{alg}(X, x)$ ) associated to  $\mathbf{Mod}_c(D_{S/k}, s)$  (resp.  $\mathbf{Mod}_c(D_{X/S/k}, f, s)$ ). Furthermore, the canonical functor  $\mathbf{Mod}_c(D_{S/k}) \rightarrow \mathbf{Mod}_c(D_{S/k}, s)$  (resp.  $\mathbf{Mod}_c(D_{X/k}) \rightarrow \mathbf{Mod}_c(D_{X/S/k}, f, s)$ ) is fully faithful and stable under taking subquotients. Thus we get a canonical surjection  $\hat{\pi}^{alg}(S, s) \twoheadrightarrow \pi^{alg}(S, s)$  (resp.  $\hat{\pi}^{alg}(X, x) \twoheadrightarrow \pi^{alg}(X, x)$ ).*

Using the results in the previous sections and apply our above lemma to  $\mathbf{Mod}_c(D_{X/S/k}, f, s)$  and  $\mathbf{Mod}_c(D_{S/k}, s)$  we get:

**Theorem 3.11.** *The homotopy sequence*

$$\pi^{alg}(X, x) \rightarrow \hat{\pi}^{alg}(X, x) \rightarrow \hat{\pi}^{alg}(S, s) \rightarrow 1$$

is exact. And one has a commutative diagram:

$$\begin{array}{ccccccc} \pi^{\text{alg}}(X_s, x) & \longrightarrow & \hat{\pi}^{\text{alg}}(X, x) & \longrightarrow & \hat{\pi}^{\text{alg}}(S, s) & \longrightarrow & 1 \\ \parallel & & \downarrow & & \downarrow & & \\ \pi^{\text{alg}}(X_s, x) & \longrightarrow & \pi^{\text{alg}}(X, x) & \longrightarrow & \pi^{\text{alg}}(S, s) & \longrightarrow & 1 \end{array}$$

**Theorem 3.12.** *Under the hypothesis in the beginning of this subsection, the homotopy sequence*

$$\pi^{\text{alg}}(X_s, x) \rightarrow \pi^{\text{alg}}(X, x) \rightarrow \pi^{\text{alg}}(S, s) \rightarrow 1$$

is exact.

*Proof.* From the surjectivity of  $\hat{\pi}^{\text{alg}}(X, x) \twoheadrightarrow \pi^{\text{alg}}(X, x)$  we know that the image of  $\pi^{\text{alg}}(X_s, x)$  is a normal subgroup of  $\pi^{\text{alg}}(X, x)$ . Then we take  $G := \text{Coker}(\pi^{\text{alg}}(X_s, x) \rightarrow \pi^{\text{alg}}(X, x))$ , and we get a surjective map of group schemes  $G \twoheadrightarrow \pi^{\text{alg}}(S, s)$ . From condition (a) we know the functor

$$\text{Rep}_k(\pi^{\text{alg}}(S, s)) \rightarrow \text{Rep}_k(G)$$

is essentially surjective, while the surjectivity of  $G \twoheadrightarrow \pi^{\text{alg}}(S, s)$  tells us that

$$\text{Rep}_k(\pi^{\text{alg}}(S, s)) \rightarrow \text{Rep}_k(G)$$

is an equivalence of categories. This finishes the proof.  $\square$

**3.3. The general case.** In this subsection we come to the general case:  $f : X \rightarrow S$  be a proper smooth morphism between two smooth connected schemes of finite type over a field  $k$  of characteristic 0 with geometrically connected fibres,  $x \in X(k)$ ,  $s \in S(k)$  and  $f(x) = s$ .

**Proposition 3.13.** *If  $k \subseteq k'$  is a field extension,  $f', X', S', x', s'$  are the corresponding morphism, schemes, points obtained by base change, and if the sequence*

$$\pi^{\text{alg}}(X'_{s'}, x') \rightarrow \pi^{\text{alg}}(X', x') \rightarrow \pi^{\text{alg}}(S', s') \rightarrow 1$$

is exact as  $k'$ -group schemes, then the sequence of  $k$ -group schemes

$$\pi^{\text{alg}}(X_s, x) \rightarrow \pi^{\text{alg}}(X, x) \rightarrow \pi^{\text{alg}}(S, s) \rightarrow 1$$

is also exact.

*Proof.* Let  $\mathfrak{C}(X')$  be the full subcategory of  $\mathbf{Mod}_c(D_{X'/k'})$  whose objects, after being pushed forward along the projection  $X' \rightarrow X$ , are the inductive limits of their coherent subobjects (i.e. subobjects belong to  $\mathbf{Mod}_c(D_{X/k})$ ). This  $\mathfrak{C}(X')$  is a Tannakian subcategory and its Tannakian group is precisely  $\pi^{\text{alg}}(X, x) \times_k k'$  [De1][10.38, 10.41]. But it is clear that this full subcategory is also stable under taking subquotients. Thus the canonical map  $\pi^{\text{alg}}(X', x') \rightarrow \pi^{\text{alg}}(X, x) \times_k k'$  is surjective. The same argument applies to  $X_s$  and  $S$ .

Hence we get a commutative diagram with the first row being exact

$$\begin{array}{ccccccc}
\pi^{\text{alg}}(X'_{s'}, x') & \xrightarrow{a'} & \pi^{\text{alg}}(X', x') & \longrightarrow & \pi^{\text{alg}}(S', s') & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \\
\pi^{\text{alg}}(X_s, x) \times_k k' & \xrightarrow{a} & \pi^{\text{alg}}(X, x) \times_k k' & \longrightarrow & \pi^{\text{alg}}(S, s) \times_k k' & \longrightarrow & 1
\end{array}$$

Since the image of  $a'$  is normal, so the image of  $a$  is also normal. Hence the image of  $\pi^{\text{alg}}(X_s, x) \rightarrow \pi^{\text{alg}}(X, x)$  is normal. Using the same argument employed in Theorem 3.12 we conclude the proof of this proposition.  $\square$

**Theorem 3.14.** *Let  $f : X \rightarrow S$  be a proper smooth morphism between two smooth connected schemes of finite type over a field  $k$  of characteristic 0 with geometrically connected fibres,  $x \in X(k)$ ,  $s \in S(k)$  and  $f(x) = s$ . Then the homotopy sequence*

$$\pi^{\text{alg}}(X_s, x) \rightarrow \pi^{\text{alg}}(X, x) \rightarrow \pi^{\text{alg}}(S, s) \rightarrow 1$$

*is exact.*

*Proof.* Since condition (a) (b) and surjectivity have been proved in §3.1, so we only need to check condition (c). But for any object  $(E, \nabla_E) \in \mathbf{Mod}_c(D_{X/k})$  and any morphism

$$\delta : (F', \nabla_{F'}) \subseteq (E, \nabla_E)|_{X_s/k} \in \mathbf{Mod}_c(D_{X_s/k}),$$

there is a finitely generated field over  $\mathbb{Q}$  on which all these objects  $(X, S, (E, \nabla_E), \dots)$  and morphisms  $(f, x, s, \delta, \dots)$  are defined. So we can reduce our problem to the case when  $k$  is a finitely generated field over  $\mathbb{Q}$ . But in light of the previous proposition we can assume our field  $k$  is actually  $\mathbb{C}$ .

Let  $K$  be the algebraic closure of the function field of  $S$ . Since  $K$  and  $\mathbb{C}$  have the same transcendental degree over  $\mathbb{Q}$ , they are isomorphic as fields. Now  $\eta : \text{Spec}(K) \hookrightarrow S$  is a geometric generic point, so by the discussion in §3.2 the sequence

$$\pi^{\text{alg}}(X_\eta, \eta') \rightarrow \pi^{\text{alg}}(X_K, \eta') \rightarrow \pi^{\text{alg}}(S_K, \eta) \rightarrow 1$$

is exact (where  $X_K$  (resp.  $S_K$ ) is the base change of  $X$  (resp.  $S$ ) from  $k$  to  $K$ , and  $\eta'$  is any chosen  $K$ -rational point of  $X_K$  above  $\eta$ ). From the lemma below we get a commutative diagram of  $K$ -group schemes

$$\begin{array}{ccccccc}
\pi^{\text{alg}}(X_\eta, \eta') & \longrightarrow & \pi^{\text{alg}}(X_K, \eta') & \longrightarrow & \pi^{\text{alg}}(S_K, \eta) & \longrightarrow & 1 \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
\pi^{\text{alg}}(X_{s_K}, x_K) & \longrightarrow & \pi^{\text{alg}}(X_K, x_K) & \longrightarrow & \pi^{\text{alg}}(S_K, s_K) & \longrightarrow & 1
\end{array}$$

Thus the last row is exact. Then we can conclude our theorem by our previous proposition.  $\square$

**Lemma 3.15.** *If  $f : X \rightarrow S$  is a smooth proper morphism between two smooth quasi-compact geometrically connected  $\mathbb{C}$ -schemes with geometrically connected fibres,  $x, x'$  and*

$s, s'$  are  $\mathbb{C}$ -rational points of  $X$  and  $S$  respectively with  $f(x) = s$  and  $f(x') = s'$ , then there exists a commutative diagramme of  $\mathbb{C}$ -group schemes

$$\begin{array}{ccccc} \pi^{alg}(X_{s'}, x') & \longrightarrow & \pi^{alg}(X, x') & \longrightarrow & \pi^{alg}(S, s') \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \pi^{alg}(X_s, x) & \longrightarrow & \pi^{alg}(X, x) & \longrightarrow & \pi^{alg}(S, s) \end{array}$$

*Proof.* From the sequences of  $\mathbb{C}$ -schemes:

$$X_s \rightarrow X \xrightarrow{f} S \quad \text{and} \quad X_{s'} \rightarrow X \xrightarrow{f} S$$

one gets sequences of analytic spaces:

$$X_s^{an} \rightarrow X^{an} \xrightarrow{f^{an}} S^{an} \quad \text{and} \quad X_{s'}^{an} \rightarrow X^{an} \xrightarrow{f^{an}} S^{an},$$

where  $X_s^{an}$  (resp.  $X_{s'}^{an}$ ) is still the fibre of  $s \in S^{an}$  (resp.  $s' \in S^{an}$ ) under  $f^{an}$ , since the functor  $-^{an}$  commutes with fibre product [SGA1][Exposé XII, 1.2]. Now applying the first homotopy functor (in topology) to these analytic spaces one gets a commutative diagram:

$$\begin{array}{ccccc} \pi_1^{\text{top}}(X_{s'}^{an}, x') & \longrightarrow & \pi_1^{\text{top}}(X^{an}, x') & \longrightarrow & \pi_1^{\text{top}}(S^{an}, s') \\ & & \downarrow \cong & & \downarrow \cong \\ \pi_1^{\text{top}}(X_s^{an}, x) & \longrightarrow & \pi_1^{\text{top}}(X^{an}, x) & \longrightarrow & \pi_1^{\text{top}}(S^{an}, s) \end{array}$$

In fact by carefully choosing a path between  $x$  and  $x'$ , there exists a group isomorphism

$$\pi_1^{\text{top}}(X_s^{an}, x) \xrightarrow{\cong} \pi_1^{\text{top}}(X_{s'}^{an}, x')$$

making the above diagramme commutative.

To show this one first defines a subset  $Z \subseteq S^{an}$  consists of points  $t \in S^{an}$  which admits a point  $y \in X^{an}$ , a path  $\alpha$  between  $x$  and  $y$ , and an isomorphism

$$\pi_1^{\text{top}}(X_s^{an}, x) \xrightarrow{\cong} \pi_1^{\text{top}}(X_t^{an}, y)$$

making the diagram

$$\begin{array}{ccc} \pi_1^{\text{top}}(X_s^{an}, x) & \longrightarrow & \pi_1^{\text{top}}(X^{an}, x) \\ \downarrow \cong & & \downarrow \alpha \\ \pi_1^{\text{top}}(X_t^{an}, y) & \longrightarrow & \pi_1^{\text{top}}(X^{an}, y) \end{array}$$

commutative.  $Z$  is both open and closed, since for any  $t \in S^{an}$  by Ehresmann's theorem ( $f^{an}$  is proper smooth by [SGA1][Exposé XII, proposition 3.1 et proposition 3.2]) one knows that in a neighborhood  $U$  of  $t \in S^{an}$   $f^{an-1}(U)$  is isomorphic to  $X_t^{an} \times S^{an}$  as a topological

space. Thus for any  $t' \in U$  one gets a commutative diagram

$$\begin{array}{ccc} \pi_1^{\text{top}}(X_t^{\text{an}}, y) & \longrightarrow & \pi_1^{\text{top}}(X^{\text{an}}, y) \\ \downarrow \cong & & \downarrow \cong \\ \pi_1^{\text{top}}(X_{t'}^{\text{an}}, y') & \longrightarrow & \pi_1^{\text{top}}(X^{\text{an}}, y') \end{array}$$

by choosing any points  $y, y' \in f^{\text{an}-1}(U)$  and any path between them (inside  $f^{\text{an}-1}(U)$ ). Hence  $t \in Z$  if and only if  $t' \in Z$ . This shows that  $Z$  is both open and closed. On the other hand, we know that  $S^{\text{an}}$  is connected ([SGA1][Exposé XII, proposition 2.4]). Thus  $Z = S^{\text{an}}$  and hence  $s' \in Z$ .

Now let us denote the category of integrable analytic connections on  $X^{\text{an}}$  and  $X_s^{\text{an}}, X_{s'}^{\text{an}}$  by  $\text{Conn}(X^{\text{an}})$  and  $\text{Conn}(X_s^{\text{an}}), \text{Conn}(X_{s'}^{\text{an}})$  respectively. By Riemann-Hilbert correspondence one has a 2-commutative diagram of neutral Tannakian categories (i.e. the  $k$ -linear tensor functors in the diagram also respect the fibres functors):

$$\begin{array}{ccc} & \text{Conn}(X_s^{\text{an}}) \xrightarrow{\cong} \text{Rep}_{\mathbb{C}}(\pi_1^{\text{top}}(X_s^{\text{an}}, x)) & \\ \nearrow \lambda & \downarrow \cong & \downarrow \cong \\ \text{Conn}(X^{\text{an}}) & & \\ \searrow \lambda' & \text{Conn}(X_{s'}^{\text{an}}) \xrightarrow{\cong} \text{Rep}_{\mathbb{C}}(\pi_1^{\text{top}}(X_{s'}^{\text{an}}, x')) & \end{array}$$

Let us set  $\iota$  to be the canonical functor  $\text{Mod}_c(D_{X/\mathbb{C}}) \rightarrow \text{Conn}(X^{\text{an}})$  sending an integrable algebraic connection on  $X$  to an integrable analytic connection on  $X^{\text{an}}$ . This functor gives a 2-commutative diagram of neutral Tannakian categories:

$$\begin{array}{ccccc} & & \text{Mod}_c(D_{X_s/\mathbb{C}}) \xrightarrow{\cong} \text{Conn}(X_s^{\text{an}}) & & \\ & \nearrow \tilde{\lambda} & & \nearrow \lambda & \\ \text{Mod}_c(D_{S/\mathbb{C}}) & \longrightarrow & \text{Mod}_c(D_{X/\mathbb{C}}) \xrightarrow{\iota} \text{Conn}(X^{\text{an}}) & & \downarrow \cong \\ & \searrow \tilde{\lambda}' & & \searrow \lambda' & \\ & & \text{Mod}_c(D_{X_{s'}/\mathbb{C}}) \xrightarrow{\cong} \text{Conn}(X_{s'}^{\text{an}}) & & \end{array}$$

Now applying Tannakian duality we conclude the proof of our lemma.  $\square$

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